ZETA FUNCTION IDENTITIES OVER THE NON-POWERS

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Abstract. Transforming the summation terms of the zeta function by the geometric series identity into terms over the non-power integers yields interesting identities for the zeta function. This paper is currently missing lots of bits to be restored eventually - a work in progress

1. Introduction

Every integer > 1 is either a perfect power or it isn't. Let $\mathbb{V}$ be the non-powers [1] and $\mathbb{W}$ be the powers [2]. Obviously, $1 \cup \mathbb{V} \cup \mathbb{W} = \mathbb{Z}^+$ and $\mathbb{V} \cap \mathbb{W} = \emptyset$.

The prime factorization of any given non-power, $v = p_1^{j_1} \cdot p_2^{j_2} \cdots p_r^{j_r}$, is by definition, such that $\gcd(j_1, j_2, \ldots, j_r) = 1$. For any given v, the numbers, $v^s \forall s > 1$ (integer s), define a unique subset of unique elements of $\mathbb{W}$.

To see this is so, take any power, w, and note it has only one possible non-power, v, as its base. Taking any v and raising it to each integer power, $s > 1$, yields a subset of $\mathbb{W}$ corresponding to all powers with non-power base v, which are obviously unique from each other and from all other remaining elements of $\mathbb{W}$. Thus, over $\mathbb{V}$, over all $s > 1$, all elements of $\mathbb{W}$ are accounted for completely and uniquely.

The prime factorization of a power is then: $w = v^s = (p_1^{j_1} \cdot p_2^{j_2} \cdots p_r^{j_r})^s$ where $s > 1$, because $\gcd(j_1, \ldots, j_r) = 1$ is still required to ensure the base, v, always belongs to $\mathbb{V}$. Each unique element of $\mathbb{W}$ is then in one to one correspondence with the unique pairs (v, s).

These transform lemmas follow:

Lemma 1.

$$\sum_{\mathbb{V}} \sum_{s=2}^{\infty} f(v^s) = \sum_{\mathbb{W}} f(w)$$

Lemma 2.

$$\sum_{\mathbb{V}} \sum_{s=1}^{\infty} f(v^s) = \sum_{k=2}^{\infty} f(k) = \sum_{\mathbb{V}} f(v) + \sum_{\mathbb{W}} f(w)$$

2. The zeta function by the non-powers

Lemma(1) leads to, and proves, the equivalency:

Theorem 1.

$$\zeta(z) = 1 + \sum_{\mathbb{V}} \frac{1}{v^z - 1} \tag{2.1}$$

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Proof. To derive the equivalence, use Lemma(1) and the geometric series identity $\sum_{s=1}^{\infty}\frac{1}{x^s}=\frac{1}{x-1}$, to split it and fiddle with it thusly:

$$\zeta(z)=\sum_{k=1}^{\infty}\frac{1}{k^z}=1+\sum_{\mathbb{V}}\frac{1}{v^z}+\sum_{\mathbb{W}}\frac{1}{w^z} \tag{2.2}$$

$$\zeta(z)=1+\sum_{\mathbb{V}}\frac{1}{v^z}+\sum_{\mathbb{V}}\sum_{s=2}^{\infty}\frac{1}{v^{sz}}$$

$$\zeta(z)=1+\sum_{\mathbb{V}}\sum_{s=1}^{\infty}\frac{1}{v^{sz}}$$

$$\zeta(z)=1+\sum_{\mathbb{V}}\frac{1}{v^z-1}$$

Or, use Lemma (2) to do it in 2 steps:

$$\zeta(z)=\sum_{k=1}^{\infty}\frac{1}{k^z}=1+\sum_{\mathbb{V}}\sum_{s=1}^{\infty}\frac{1}{v^{sz}} \tag{2.3}$$

$$\zeta(z)=1+\sum_{\mathbb{V}}\frac{1}{v^z-1}$$

□

Consider the infinite set of summation terms on each side of:

$$\sum_{k=1}^{\infty}\frac{1}{k^z}=1+\sum_{\mathbb{V}}\frac{1}{v^z-1}$$

Being equivalent, each side has the same radius of convergence and, when convergent, the same functional value for any given z. Although only formally equivalent for $z\leq 1$, it does seem it can be evaluated at $z=1$ where the equivalence when divergent can be demonstrated directly. Substitute *Goldbach's* well-known result, $\sum_{\mathbb{W}}\frac{1}{w-1}=1$ [*per Euler* [3]], into Eq.(2.1) with $z=1$ to express:

$$\zeta(1)=\sum_{k=1}^{\infty}\frac{1}{k}=\sum_{\mathbb{W}}\frac{1}{w-1}+\sum_{\mathbb{V}}\frac{1}{v-1} \tag{2.4}$$

where each side then matches term by term infinitely.

3. Requirements of the zeta by non-powers equivalence

Eq.(2.1) can be rewritten

$$\zeta(z)=1+\sum_{\mathbb{V}}\frac{1}{2(v^{z/2}-1)}-\sum_{\mathbb{V}}\frac{1}{2(v^{z/2}+1)} \tag{3.1}$$

And Eq.(2.1) is an identity for the zeta function itself, thus

$$\zeta(z)=1+\frac{\zeta(z/2)-1}{2}-\sum_{\mathbb{V}}\frac{1}{2(v^{z/2}+1)} \tag{3.2}$$

so that,

$$\sum_{\mathbb{V}} \frac{1}{v^{z/2}+1} = 1 + \zeta(z/2) - 2\zeta(z) \tag{3.3}$$

Theorem 2. *Where it converges*

$$\sum_{\mathbb{V}} \frac{v^z+2}{v^{2z}(v^z+1)} = \sum_{\mathbb{W}} \frac{w^z-2}{w^{2z}} \tag{3.4}$$

Proof. Convert the zetas on the *rhs* of of Eq.(**??**) to their standard sum form and solve

$$\sum_{\mathbb{V}} \frac{1}{v^z+1} = \sum_{k=2}^{\infty} \frac{1}{k^z}(1-\frac{2}{k^z})$$

$$\sum_{\mathbb{V}} \frac{1}{v^z+1} = \sum_{\mathbb{V}} \frac{1}{v^z}(1-\frac{2}{v^z}) + \sum_{\mathbb{W}} \frac{1}{w^z}(1-\frac{2}{w^z})$$

$$\sum_{\mathbb{V}} \frac{v^z+2}{v^{2z}(v^z+1)} = \sum_{\mathbb{W}} \frac{w^z-2}{w^{2z}}$$

□

4. A telescoping zeta-function sum

Obviously, the splitting performed to get Eq.(3.1) could continue indefinitely.

$$\zeta(z) = 1 + \sum_{\mathbb{V}} \frac{1}{4(v^{z/4}-1)} - \sum_{\mathbb{V}} \frac{1}{4(v^{z/4}+1)} - \sum_{\mathbb{V}} \frac{1}{2(v^{z/2}+1)}$$

$$\zeta(z) = 1 + \sum_{\mathbb{V}} \frac{1}{8(v^{z/8}-1)} - \sum_{\mathbb{V}} \frac{1}{8(v^{z/8}+1)} - \sum_{\mathbb{V}} \frac{1}{4(v^{z/4}+1)} - \sum_{\mathbb{V}} \frac{1}{2(v^{z/2}+1)}$$

$$\vdots$$

Transforming in the other direction leads to a generalization which bears further on the argument of Conjecture(**??**) and may be a good starting point from which to seek an alternate demonstration of a proof.

Theorem 3. *For any complex z and integers $\{a, b \mid a \leq b\}$, the following identity holds:*

$$2^a\zeta(2^{a-1}z) - 2^{b+1}\zeta(2^b z) = \sum_{j=a}^{b} 2^j (\sum_{\mathbb{V}} \frac{1}{v^{2^{j-1}z}+1} - 1) \tag{4.1}$$

Proof. Begin with multiple occurrences of identities from Eq.(**??**) as follows:

$$\vdots$$

$$\zeta(z/8) - 2\zeta(z/4) = -1 + \sum_{\mathbb{V}} \frac{1}{v^{z/8}+1}$$

$$\zeta(z/4) - 2\zeta(z/2) = -1 + \sum_{\mathbb{V}} \frac{1}{v^{z/4}+1}$$

$$\zeta(z/2) - 2\zeta(z) = -1 + \sum_{\mathbb{V}} \frac{1}{v^{z/2}+1}$$

$$\zeta(z) - 2\zeta(2z) = -1 + \sum_{\mathbb{V}} \frac{1}{v^{z}+1}$$

$$\zeta(2z) - 2\zeta(4z) = -1 + \sum_{\mathbb{V}} \frac{1}{v^{2z}+1}$$

$$\vdots$$

and let the middle line correspond to $j = 0$ and multiply each equality by 2^j

$$\vdots \qquad j = a$$

$$\zeta(z/8)/4 - \zeta(z/4)/2 = -1/4 + \sum_{\mathbb{V}} \frac{1}{4(v^{z/8}+1)} \qquad j = -2$$

$$\zeta(z/4)/2 - \zeta(z/2) = -1/2 + \sum_{\mathbb{V}} \frac{1}{2(v^{z/4}+1)} \qquad j = -1$$

$$\zeta(z/2) - 2\zeta(z) = -1 + \sum_{\mathbb{V}} \frac{1}{v^{z/2}+1} \qquad j = 0$$

$$2\zeta(z) - 4\zeta(2z) = -2 + \sum_{\mathbb{V}} \frac{2}{v^{z}+1} \qquad j = 1$$

$$4\zeta(2z) - 8\zeta(4z) = -4 + \sum_{\mathbb{V}} \frac{4}{v^{2z}+1} \qquad j = 2$$

$$\vdots \qquad j = b$$

When summed over j from a to b, the *lhs* telescopes to yield Eq.(4.1), completing the proof. □

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