THE NUMBER THEORETIC NATURE OF THE MULTIPLICITY OF $\sum_{k=2}^{\infty}\sum_{s=2}^{\infty} k^{-s} = 1$

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Abstract. This is a working paper on Imathination.org

1. Introduction

The historical accounts of the double-sum equality $\sum_{k=2}^{\infty}\sum_{s=2}^{\infty} k^{-s} = 1$ are tangled, in part because that double sum is so many other things, a property stemming from its equivalence to unity. Finding an authoritative reference that gets the story right and clears the air of misconceptions, was a doomed effort for me. Sometimes attributed to *Leibniz* and/or *Huygens*, or even *Goldbach* via *Euler*, but the earliest variant I could find was *Mengoli's* sum of the inverses of the triangular numbers, $\sum_{k=1}^{\infty} T_k^{-1} = 2$, where $T_k = k(k+1)/2$.

The geometric series identity, which, for each given k, transforms the inner sum $\sum_{s=2}^{\infty} k^{-s}$ into $((k-1)k)^{-1}$, was known to the ancients, thus to *Mengoli*. And as such,

$$\sum_{k=2}^{\infty}((k-1)k)^{-1} = \sum_{k=1}^{\infty}(k(k+1))^{-1} = \sum_{k=1}^{\infty}(2T_k)^{-1} = 1$$

is trivial under the transform afforded by the, even then, well-known identity.

The terms of the double sum, in any form, can be split into many more terms or combined into a smaller number of terms an infinite number of ways. Some are more elegant than most of the possibilities. Summing over all k for each s yields $\sum_{s=2}^{\infty}(\zeta(s)-1) = 1$, which I have seen attributed to Goldbach, but it's just a notational/combinatorial variant.

Some of the *Euler/Goldbach* confusion is probably due to *Goldbach's* similar looking result, as transmitted by *Euler*, $\sum_{k,s>1}(k^s-1)^{-1} = 1$, where each summation term is unique and their set is in one to one correspondance with the perfect powers, often noted as 'without multiplicity' or 'without duplicity'.[*Euler* [1]]. This too, is equivalent to the double sum, as anything equal to unity must be. It can be obtained from the mulitiplicitous double sum directly using the same ancient transform and a little recombining of terms.

But what of the nature of the multiplicity? What can be said of it? When I looked, I found the kind of thing that should have been found before, then I found it hadn't, so here it is now.

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2. Infinite sum transforms by the geometric series identity

Every integer > 1 is either a perfect power or it isn't [*Sloane's* [2]]. Let $\mathbb{V}$ be the non-powers and $\mathbb{W}$ be the powers. Obviously, $1 \cup \mathbb{V} \cup \mathbb{W} = \mathbb{Z}^+$ and $\mathbb{V} \cap \mathbb{W} = \emptyset$.

The prime factorization of any given non-power, $v = p_1^{j_1} \cdot p_2^{j_2} \cdots p_r^{j_r}$, is by definition, such that $\gcd(j_1, j_2, \ldots, j_r) = 1$. For any given v, the numbers, $v^s \forall s > 1$ (integer s), define a unique subset of unique elements of $\mathbb{W}$.

To see this is so, take any power, w, and note it has only one possible non-power, v, as its base. Taking any v and raising it to each integer power, $s > 1$, yields a subset of $\mathbb{W}$ corresponding to all powers with non-power base v, which are obviously unique from each other and from all other remaining elements of $\mathbb{W}$. Thus, over $\mathbb{V}$, over all $s > 1$, all elements of $\mathbb{W}$ are accounted for completely and uniquely.

The prime factorization of a power is then: $w = v^s = (p_1^{j_1} \cdot p_2^{j_2} \cdots p_r^{j_r})^s$ where $s > 1$, because $\gcd(j_1, \ldots, j_r) = 1$ is still required to ensure the base, v, always belongs to $\mathbb{V}$. Each unique element of $\mathbb{W}$ is then in one to one correspondence with the unique pairs (v, s).

These transform lemmas follow:

Lemma 1.

$$\sum_{\mathbb{V}} \sum_{s=2}^{\infty} f(v^s) = \sum_{\mathbb{W}} f(w)$$

Lemma 2.

$$\sum_{\mathbb{V}} \sum_{s=1}^{\infty} f(v^s) = \sum_{k=2}^{\infty} f(k) = \sum_{\mathbb{V}} f(v) + \sum_{\mathbb{W}} f(w)$$

3. Elegance for Goldbach's sum

Goldbach's result, $\sum_{\mathbb{W}} (w-1)^{-1} = 1$, has an elegant proof using the 'essentially known to Mengoli' double sum as the starting point:

$$\sum_{k=2}^{\infty} \sum_{s=2}^{\infty} \frac{1}{k^s} = 1$$

By Lemma 2

$$\sum_{\mathbb{W}} \sum_{s=2}^{\infty} \frac{1}{w^s} + \sum_{\mathbb{V}} \sum_{s=2}^{\infty} \frac{1}{v^s} = 1$$

By Lemma 1

$$\sum_{\mathbb{W}} \sum_{s=2}^{\infty} \frac{1}{w^s} + \sum_{\mathbb{W}} \frac{1}{w} = 1$$

$$\sum_{\mathbb{W}} \sum_{s=1}^{\infty} \frac{1}{w^s} = 1$$

By the geometric series identity

$$\sum_{\mathbb{W}} \frac{1}{w-1} = 1$$

The above proof of *Goldbach's* result, using a split into the powers and non-powers, is not unknown. A discussion is found in [Viader et al [3], the folks who translated [1]].

4. The number theoretic nature of the multiplicity of $\sum_{k=2}^{\infty}\sum_{s=2}^{\infty} k^{-s}=1$

Theorem 1. *The multiplicity, or duplication, of perfect power inverses among the terms of* $\sum_{k=2}^{\infty}\sum_{s=2}^{\infty} k^{-s}=1$ *is described over the non-powers by the equality:*

$$\sum_{\mathbb{V}}\sum_{s=2}^{\infty}\frac{\tau(s)-1}{v^s}=1 \tag{4.1}$$

where $\tau(s)$ *is the number of divisors of* s.

Proof.

$$\sum_{k=2}^{\infty}\sum_{r=2}^{\infty}\frac{1}{k^r}=1$$

By Lemma 2

$$\sum_{\mathbb{V}}\sum_{r=2}^{\infty}\frac{1}{v^r}+\sum_{\mathbb{W}}\sum_{r=2}^{\infty}\frac{1}{w^r}=1$$

By Lemma 1

$$\sum_{\mathbb{V}}\sum_{r=2}^{\infty}\frac{1}{v^r}+\sum_{\mathbb{V}}\sum_{t=2}^{\infty}\sum_{r=2}^{\infty}\frac{1}{v^{tr}}=1 \tag{4.2}$$

The leftmost, double-sum term on the *lhs* of Eq.(4.2), includes a single, unique summation term for each power, which is obvious from the Lemma (1) transform to $\sum_{\mathbb{W}} 1/w$. Thus, Eq.(4.2) is a split of the summation terms into the 2 subsums, *'unique terms'* + *'replicated terms'* $= 1$.

The double sum also corresponds to the $t=1$ case of the triple sum, thus:

$$\sum_{\mathbb{V}}\sum_{t=1}^{\infty}\sum_{r=2}^{\infty}\frac{1}{v^{tr}}=1 \tag{4.3}$$

Which again, is just a notational/combinatorial variant of the original double sum, completely equivalent to any of its other variants.

The value of the exponent tr in Eq.(4.3) must always be a positive integer >1. So, a third integer, $s>1$, can be used to index the sum if it is known how many times $rt=s$ over t, over r in Eq.(4.3). To retain unique factorization in integers, (t,r) must be one of the unique pairings of 2 divisors of s, such that $d\cdot d'=s$.

Over all divisors of any given s, the 2 sets of elements $\{d\}$ and $\{d'\}$ become the same set, forming in opposite order to each other, including, if and only when s is a square, one instance of $d=d'$. In all other cases, the divisors must be paired such that $d<\sqrt{s}$ and $d'>\sqrt{s}$ or vice-versa. The count of distinct pairings is of course, $\tau(s)$ in all cases.

Since r starts at 2 in Eq.(4.3) however, there is no summation term corresponding to the pair $t=s$, $r=1$ for any s, thus one pairing of divisors is missing for each s. Therefore, over each v and each s, the sum of Eq.(4.3) will have exactly $\tau(s)-1$ terms for which $1/v^{tr}$ is identical to $1/v^s$, so reindex the inner double sum over s, replacing the double indexes, $\{t,r\}$, and multiply each unique summation term by the number of times it occurs, $(\tau(s)-1)$, completing the proof.

□

The 'unique' + 'replicated' sums of Eq.(4.2) can then be written

$$\sum_{\mathbb{W}} \frac{1}{w} + \sum_{\mathbb{V}} \sum_{s=2}^{\infty} \frac{\tau(s) - 2}{v^s} = 1 \tag{4.4}$$

which gives the equivalent expressions for the sum of the replicated terms

$$\sum_{\mathbb{V}} \sum_{t=2}^{\infty} \sum_{r=2}^{\infty} \frac{1}{v^{tr}} = \sum_{\mathbb{V}} \sum_{s=2}^{\infty} \frac{\tau(s) - 2}{v^s} = \sum_{\mathbb{W}} \frac{1}{(w-1)w} \tag{4.5}$$

The following function definition proves useful for working along these lines over $\mathbb{W}$ instead of $\mathbb{V}$. I will replace it with the accepted number theoretic function, if I discover such a thing exists, but this suffices in a pinch:

Definition 1. $\texttt{gip}(w)$ = the exponent of a perfect power's non-power base.

Simple enough, for every power, $w = v^s$, the function $\texttt{gip}(w)$ returns the value of its power, s, where it is understood that s is always w's *greatest integer power* such that the base is reduced to its associated non-power, v.

Theorem (1) and Definition (1) then provide the additional transform lemma:

Lemma 3.

$$\sum_{\mathbb{V}} \sum_{s=2}^{\infty} (\tau(s) - 1) f(v^s) = \sum_{k=2}^{\infty} \sum_{s=2}^{\infty} f(k^s) = \sum_{\mathbb{W}} (\tau(\texttt{gip}(w)) - 1) f(w)$$

Which allows Eq.(4.5) to solve to

$$\sum_{\mathbb{W}} \frac{(\tau(\texttt{gip}(w)) - 2)(w-1) - 1}{(w-1)w} = 0 \tag{4.6}$$

or, in the non-power notation where the index s is split into the 2 sets, the primes, P, and the composites, C, the following relation over the non-powers with composite and prime exponents on opposites sides of the equality is obtained

$$\sum_{V} \sum_{C} \frac{(\tau(c) - 2)(v^c - 1) - 1}{(v^c - 1)v^c} = \sum_{V} \sum_{P} \frac{1}{(v^p - 1)v^p} \tag{4.7}$$

Though a bit trickier for theoretical work and likely far from ideal from a computational standpoint, Eq.(4.7) does represent a fundamental number theoretic identity that isolates the primes on one side of the equality. The fact that it incorporates the number of divisors function makes it all the more intriguing. Further, adding $\sum_V \sum_C ((v^c - 1)v^c)^{-1}$ to each side yields the curious and beautiful form

$$\sum_{V} \sum_{C} \frac{(\tau(c) - 2)}{v^c} = \sum_{W} \frac{1}{(w-1)w} \tag{4.8}$$

.

5. Prolific Elegance Ensues

Other results follow easily from this line inquiry, including the following, where the missing pieces are left as an exercise for the moment:

$$\sum_{\mathbb{W}} \frac{1}{w+1} = 2\zeta(2) - 5/2 \tag{5.1}$$

Note, the above is already known, I find it mentioned in [Sloane's [2]], but who proved it and how, I do not know. It follows that

$$\sum_{\mathbb{W}} \frac{1}{w^2 - 1} = 7/4 - \zeta(2) \tag{5.2}$$

and

$$\sum_{\mathbb{W}} \frac{w}{w^2 - 1} = \zeta(2) - 3/4 \tag{5.3}$$

where the 2 equations above, of course, sum to unity.

In [4], I prove the zeta function by the non-powers equivalence

$$1 + \sum_{\mathbb{V}} \frac{1}{v^z - 1} = \zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} \tag{5.4}$$

which at $z = 1$ yields the non-power equivalent of the harmonic number, giving the following identity after castration (cut the 1 off) of each side.

$$\sum_{\mathbb{V}} \frac{1}{v - 1} = \sum_{k=2}^{\infty} \frac{1}{k} \tag{5.5}$$

Subtract $\sum_{\mathbb{V}} \frac{1}{v}$ from each side of Eq.(5.5) to get:

$$\sum_{\mathbb{V}} \frac{1}{(v-1)v} = \sum_{\mathbb{W}} \frac{1}{w} \tag{5.6}$$

which turns out to be the $z = 1$ case of the relation

$$\sum_{\mathbb{V}} \frac{1}{(v^z - 1)v^z} = \sum_{\mathbb{W}} \frac{1}{w^z} \tag{5.7}$$

over complex z. The 'complex harmonic sum of the powers' is equivalent to the 'sum of the inverse complex-form pronics'. I once read a note on a Sloane's sequence that 'promics' may be the more correct term for the numbers $2T_k$, the doubles of the triangular numbers.

Now, adding $\sum_{\mathbb{W}} \frac{1}{(w^z-1)w^z}$ to each side yields:

$$\sum_{k=2}^{\infty} \frac{1}{(k^z - 1)k^z} = \sum_{\mathbb{W}} \frac{1}{w^z - 1} \tag{5.8}$$

and of course we already know

$$\sum_{k=2}^{\infty} \frac{1}{(k-1)k} = 1 = \sum_{\mathbb{W}} \frac{1}{w - 1}$$

so a good question is, what happens to the value of unity in the center as the identity is taken off the pole and over complex z? Eq.(5.4) provides the form:

$$\sum_{k=2}^{\infty} \frac{1}{k^z} = \zeta(z) - 1 = \sum_{\mathbb{V}} \frac{1}{v^z - 1}$$

with the castrated zeta function in the center.

Now, de-castrate (add 1) to each side of Eq.(5.8) to get

$$1 + \sum_{k=2}^{\infty} \frac{1}{(k^z - 1)k^z} = 1 + \sum_{\mathbb{W}} \frac{1}{w^z - 1}$$

how hauntingly like the zeta function

$$1 + \sum_{k=2}^{\infty} \frac{1}{k^z} = 1 + \sum_{\mathbb{V}} \frac{1}{v^z - 1}$$

it is. Which begs the question, is there a paralell function $\zeta_?(z)$ such that

$$\sum_{k=2}^{\infty} \frac{1}{(k^z - 1)k^z} = \zeta_?(z) - 1 = \sum_{\mathbb{W}} \frac{1}{w^z - 1}$$

or, is there an accessible generalization that carries these relations to ever higher dimensions/powers. Consider, for example, the equality derived in [4]

$$\sum_{\mathbb{V}} \frac{v^z + 2}{v^{2z}(v^z + 1)} = \sum_{\mathbb{W}} \frac{w^z - 2}{w^{2z}}$$

More to follow.

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